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Torsional oscillations of a fluid sphere with a rigid boundary in a uniform magnetic field

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1. Introduction

Plumpton & Ferraro (1955) considered the torsional oscillations of an infinitely conducting sphere in a uniform magnetic field. They showed that if the fluid and magnetic viscosity were assumed to be zero in the governing differential equations, then a continuous spectrum of eigenvalues could be obtained. This

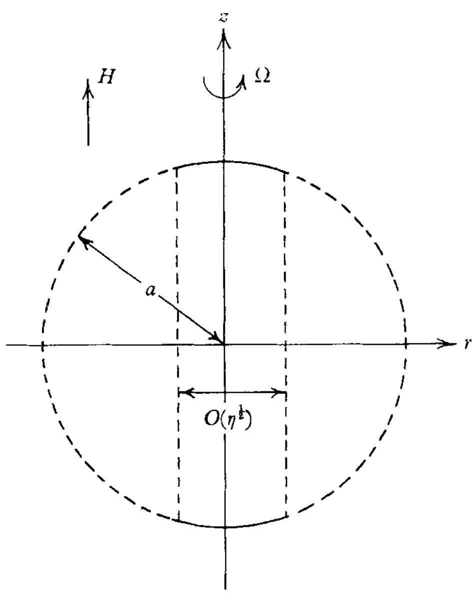

FIGURE 1. Free boundary case (Stewartson 1957). The disturbance is confined to a circular cylinder of radius $O(\sqrt{\eta})$; outside this zone the solution vanishes asymptotically.

novel feature was clarified by Stewartson (1957) when he obtained the exact solution and showed that in the correct limit of a perfect conductor the eigenvalues are discrete. Furthermore, in the limit of infinite conductivity the oscillations occur only on the axis of symmetry (figure 1).

Here we consider the case of torsional oscillations of a highly conducting, small viscosity, fluid sphere contained by a rigid wall in a uniform magnetic field. The insight obtained from the exact solution of the analogous cylindrical problem is used to formulate an asymptotic approximation of the equations governing the spherical problem. The boundary conditions are satisfied exactly and the differential equations are satisfied in an approximate manner for small values of fluid kinematic viscosity ν and magnetic viscosity η. There is a doubly infinite discrete set of eigenvalues with even and odd modes. The corresponding oscillations are confined to the immediate vicinity of the axis of symmetry with a boundary layer confined to the corresponding small region of the spherical surface. In the special case of $\eta = \nu$, which can be solved exactly, the disturbance is confined to the two points of the spherical surface along the axis of symmetry.

2. The equations and boundary conditions

It is supposed that a fluid sphere of radius a has an imposed uniform magnetic field H. If $\mathbf{v}$, $\mathbf{h}$ are the velocity and magnetic vectors in the disturbed state, Maxwell's and Euler's equation (e.m.u.) reduce to

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{h}) + \eta \nabla^2 \mathbf{h}, \tag{2.1}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{h}) \times \mathbf{h} + \rho\nu\nabla^2, \tag{2.2}$$

$$\nabla . \mathbf{v} = \nabla . \mathbf{h} = 0, \tag{2.3}$$

where $\eta = 1/(4\pi\sigma)$ is the magnetic viscosity, σ the conductivity, ν the kinematic viscosity, ρ the density (constant), P the pressure, and only the magnetic body forces are retained.

We choose a set of cylindrical co-ordinates r, ϕ, z in which the z-axis is parallel to the direction of the imposed field and the origin is the centre of the sphere. All perturbed quantities are assumed to oscillate with the same period $2\pi/\alpha$, and are small. For toroidal disturbances, symmetrical about z, we may write

$$\mathbf{v} = (0, \Omega r e^{i\alpha t}, 0), \quad \mathbf{h} = (0, hr\sqrt{(4\pi\rho)}\, e^{i\alpha t}, H), \tag{2.4}$$

where Ω and h are small functions of r and z only. The vector equations reduce to $P =$ const. and

$$i\alpha h = A\frac{\partial \Omega}{\partial z} + \eta\left(\frac{\partial^2 h}{\partial r^2} + \frac{3}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial z^2}\right), \tag{2.5}$$

$$i\alpha \Omega = A\frac{\partial h}{\partial z} + \nu\left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{3}{r}\frac{\partial \Omega}{\partial r} + \frac{\partial^2 \Omega}{\partial z^2}\right), \tag{2.6}$$

where $A = H/\sqrt{(4\pi\rho)}$ is the Alfvén velocity.

The two boundary conditions at the surface of the sphere $R = a$ are:

First, since $\mathbf{h}$ is continuous and since $\mathbf{h}$ and the potential of the external field are functions of r and z only,

$$h = 0 \quad \text{when} \quad R = a. \tag{2.7}$$

Secondly the velocity $\mathbf{v}$ must be zero on the right boundary $R = a$,

$$\Omega = 0 \quad \text{when} \quad R = a. \tag{2.8}$$

In general it is difficult to obtain non-trivial eigenvalues satisfying (2.5)–(2.8). However, we are concerned with the special case of η and ν vanishingly small. If we let η and $\nu \to 0$ in (2.5) and (2.6) we obtain,

$$i\alpha h = A(\partial\Omega/\partial z) \quad \text{and} \quad i\alpha\Omega = A(\partial h/\partial z), \tag{2.9}$$

with the solution

$$h = C(r)\cos\{\beta z + B(r)\}, \quad \Omega = iC(r)\sin\{\beta z + B(r)\}, \tag{2.10}$$

where $\alpha^2 = A^2\beta^2$.

Equations (2.9) and (2.10) are the incorrect limiting form since we obtain contradictions.

First, since $h = 0$ when $R = a$,

$$\beta(a^2 - r_0^2)^{\frac{1}{2}} + B(r_0) = (m + \tfrac{1}{2})\pi \quad \text{or} \quad C(r) = 0 \quad \text{if} \quad r \neq r_0, \tag{2.11}$$

where $r_0 < a$ and m is an integer.

Secondly, since $\Omega = 0$ when $R = a$,

$$\beta(a^2 - r_0^2)^{\frac{1}{2}} + B(r_0) = m\pi \quad \text{or} \quad C(r) = 0 \quad \text{if} \quad r \neq r_0, \tag{2.12}$$

which contradicts (2.11).

Thirdly, from (2.5) and (2.6) we require that $\partial\Omega/\partial r$ and $\partial h/\partial r$ vanish at $r = 0$ for all η, ν and z. Thus if Ω and h are not identically equal to zero, they cannot vanish everywhere on the axis of symmetry. In the limit $\eta \to 0$, $\nu \to 0$ this must also be true implying

$$C(0) \neq 0. \tag{2.13}$$

This contradicts (2.11) and (2.12) unless $r_0 = 0$ which we will presently show to be true. The contradiction between (2.11) and (2.12) may be resolved by a boundary layer in the neighbourhood of $z = \pm a$. In §3 we will solve the analogous cylindrical problem to gain insight into the spherical one.

3. Eigenvalues for the cylindrical case

In the analogous cylindrical problem we replace the fluid sphere of the previous section with a fluid cylinder defined by $r = a$, $z < |a|$ (figure 2 (*a*)). The equations and boundary conditions to be satisfied are given by (2.5) to (2.8). Hence h and Ω satisfy

$$\left[\left\{i\alpha - \eta\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\right\}\left\{i\alpha - \nu\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\right\} - A^2\frac{\partial^2}{\partial z^2}\right](h, \Omega) = 0. \tag{3.1}$$

The solution of (3.1) has the form

$$h = \phi(z)\, J_1(\mu r)/r, \tag{3.2}$$

where $J_1(\mu r)$ is the bounded Bessel function of first order and first kind, μa gives the zeros of the Bessel function satisfying $h = \Omega = 0$ at $r = a$. Thus the solution becomes

$$\phi(z) = A_1 e^{a_1 z} + A_2 e^{a_2 z} + A_3 e^{-a_1 z} + A_4 e^{-a_2 z}, \tag{3.3}$$

where

$$a_1 = \left(\frac{i\alpha(\eta+\nu) + 2\eta\nu\mu^2 + A^2 + \sqrt{\{A^4 + 4\eta\nu\mu^2 A^2 + 2i\alpha(\eta+\nu)A^2 - \alpha^2(\eta-\nu)^2\}}}{2\eta\nu}\right)^{\frac{1}{2}}, \tag{3.4}$$

and

$$a_2 = \left(\frac{i\alpha(\eta+\nu) + 2\eta\nu^2 + A^2 - \sqrt{\{A^4 + 4\eta\nu\mu^2 A^2 + 2i\alpha(\eta+\nu)A^2 - \alpha^2(\eta-\nu)^2\}}}{2\eta\nu}\right)^{\frac{1}{2}}. \tag{3.5}$$

Consequently we obtain

$$\Omega = \frac{J_1(\mu r)}{r}(\beta_1 A_1 e^{a_1 z} + \beta_2 A_2 e^{a_2 z} - \beta_1 A_3 e^{-a_1 z} - \beta_2 A_4 e^{-a_2 z}), \tag{3.6}$$

where

$$\beta_j = \frac{i\alpha + \eta(\mu^2 - a_j^2)}{Aa_j}. \tag{3.7}$$

Applying the boundary conditions to these solutions gives the following secular equation

$$e^{2a_2 a}\left[\frac{1-\exp\{-2(a_1+a_2)a\}}{1-\exp\{-2(a_1-a_2)a\}}\right] = \pm\left(\frac{\beta_1+\beta_2}{\beta_1-\beta_2}\right). \tag{3.8}$$

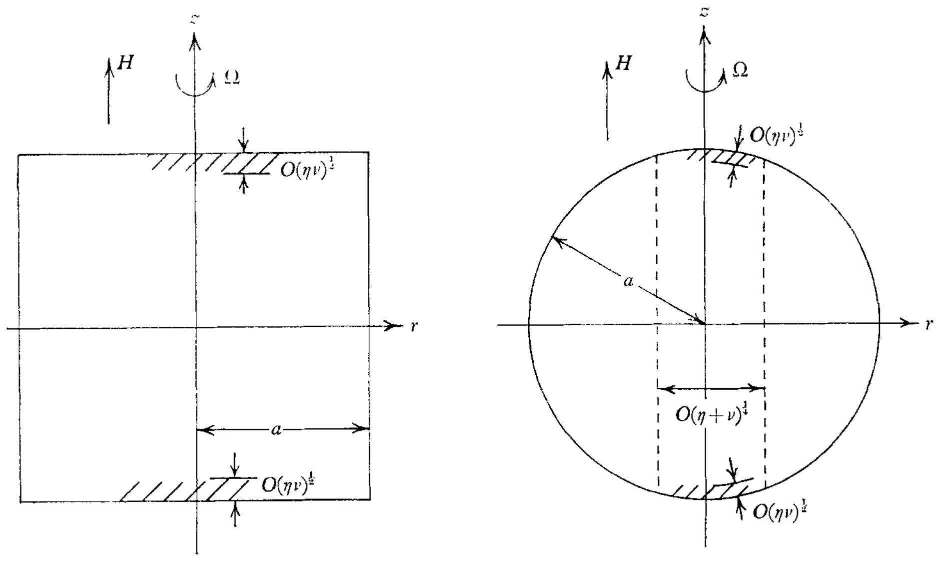

FIGURE 2. Fixed boundary case. (*a*) A boundary layer $O(\eta\nu)^{\frac{1}{2}}$ forms at both ends of the cylinder; the disturbance varies as $J_1(\mu r)/r$. (*b*) The disturbance falls off as $\exp\{-r^2/(\eta+\nu)^{\frac{1}{2}}\}$ and thus becomes negligible outside a zone of radius $O(\eta+\nu)^{\frac{1}{4}}$. Furthermore, a boundary layer $O(\eta\nu)^{\frac{1}{2}}$ is present at each pole.

Consider the case of ν and η small with $\alpha = O(1)$, $\mu = O(1)$, then

$$a_2 = \frac{i\alpha}{A}\left[1 - \frac{i(\eta+\nu)}{2A^2\alpha}(\mu^2 A^2 + \alpha^2)\right] + O(\eta\nu), \tag{3.9}$$

and

$$a_1 = \frac{A}{\sqrt{(\eta\nu)}}\left\{1 + \frac{i\alpha(\eta+\nu)}{2A^2} + O(\eta\nu)\right\}. \tag{3.10}$$

Consequently (3.8) reduces to

$$e^{2a_2 a} = \pm\left(\frac{1-K}{1+K}\right)\left\{1 + \frac{i\sqrt{(\eta\nu)}}{A^2\alpha}(A^2\mu^2 + 2\alpha^2)\right\} + O(\eta\nu), \tag{3.11}$$

where $K = \sqrt{(\nu/\eta)}$ and m is an integer. It then follows that if $K \neq 1$,

$$\alpha = \frac{A}{2a}\left\{m\pi - i\ln\left|\frac{1-K}{1+K}\right|\right\} + \tfrac{1}{2}i\,(\eta+\nu)\left\{\mu^2 + \frac{1}{4\alpha^2}\left(m\pi - i\ln\left|\frac{1-K}{1+K}\right|\right)^2\right\}$$
$$+\sqrt{(\eta\nu)}\left\{\mu^2 + \frac{1}{2a^2}\left(m\pi - i\ln\left|\frac{1-K}{1+K}\right|\right)^2\right\}\Big/\left(m\pi - i\ln\left|\frac{1-K}{1+K}\right|\right) + O(\eta+\nu)^2 \tag{3.12}$$

for finite values of μ and m.

We now demonstrate a boundary-layer approach to obtaining the eigenvalues. From the previous solution we have

$$\frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r} = O(1) \tag{3.13}$$

for μ finite.

In the interior (away from $z = \pm a$) of the fluid we let η and $\nu \to 0$ and obtain (2.9) with the solution

$$h_1 = B\sin\beta z + C\cos\beta z, \quad \Omega_1 = -iB\cos\beta z + iC\sin\beta z, \quad (3.14),\ (3.15)$$

where $\beta A = \alpha$ and C, B are functions of r only.

In the boundary layer formed near $z = \pm a$ the operator $\partial/\partial z$ is large and equations (2.5) and (2.6) reduce to

$$A\frac{\partial h}{\partial z}+\nu\frac{\partial^2 \Omega}{\partial z^2} = 0, \quad A\frac{\partial \Omega}{\partial z}+\eta\frac{\partial^2 h}{\partial z^2} = 0. \tag{3.16}$$

The appropriate solutions of these equations are

$$h_2 = D(r)\left[\exp\{(z-a)A/\surd(\eta\nu)\}-1\right] = -\Omega_2\surd(\nu/\eta) \tag{3.17}$$

at $z = a$, and

$$h_3 = E(r)\left[\exp\{-(z+a)A/\surd(\eta\nu)\}-1\right] = \Omega_3\surd(\nu/\eta) \tag{3.18}$$

at $z = -a$.

To match these solutions we require that

$$\left.\begin{aligned} h_1(a) &= h_2(-\infty), \quad \Omega_1(a) = \Omega_2(-\infty), \\ h_1(-a) &= h_3(\infty), \quad \Omega_1(-a) = \Omega_3(\infty). \end{aligned}\right\} \tag{3.19}$$

Hence for a non-trivial solution

$$(\sin\beta a - zK\cos\beta a)(\cos\beta a + zK\sin\beta a) = 0, \tag{3.20}$$

where $K = (\nu/\eta)^{\frac{1}{2}}$.

Alternatively (3.20) may be written

$$e^{2i\beta a} = \pm\left(\frac{1-K}{1+K}\right), \tag{3.21}$$

which gives

$$\alpha = \frac{m\pi A}{2a}-\frac{iA}{2a}\ln\left|\frac{1-K}{1+K}\right|, \tag{3.22}$$

where m is an integer and $K \neq 1$.

We now proceed to obtain the first-order effect of η and ν upon the eigenvalues. First, in the interior of the fluid, equation (3.1) becomes

$$\left\{\alpha^2+i\alpha(\eta+\nu)\left(\frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)+A^2\frac{\partial^2}{\partial z^2}\right\}h_1 = O(\eta\nu h_1), \tag{3.23}$$

with the solution

$$h_1 = \frac{J_1(\mu r)}{r}(B\sin\beta z + C\cos\beta z), \tag{3.24}$$

where

$$\beta = \frac{\alpha}{A}\left\{1-\frac{i(n+\nu)}{2A^2\alpha}(\mu^2A^2+\alpha^2)\right\}+O(\eta+\nu)^2. \tag{3.25}$$

From (2.5) we get

$$\Omega_1 = \frac{iJ_1(\mu r)}{r}(-B\cos\beta z + C\sin\beta z)\left\{1+\frac{i(\mu^2A^2+\alpha^2)(\nu-\eta)}{2A^2\alpha}+O(\eta+\nu)^2\right\}. \quad (3.26)$$

Secondly, in the boundary layer we have

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2}{\partial z^2}-\gamma^2\right)h = 0, \quad (3.27)$$

where $$\gamma = \frac{A}{\sqrt{(\eta\nu)}}\left\{1+\frac{i\alpha(\eta+\nu)}{2A^2}+O(\eta+\nu)^2\right\}. \quad (3.28)$$

The appropriate solution near $z = a$ is

$$ha = D[\exp\{(z-a)\gamma\}-1]+(\alpha-z)F. \quad (3.29)$$

From (2.5) we obtain the corresponding Ω, namely

$$\Omega_a = \frac{D}{K}\{1-e^{(z-a)\gamma}\}\left\{1+\frac{i\alpha(\eta\nu)}{2A^2}+O(\eta+\nu)^2\right\}+\frac{i\alpha D(a-z)}{A}+O(a-z)^2. \quad (3.30)$$

Similarly near $z = -a$ we get

$$h_{-a} = E[\exp\{-(z+a)\gamma\}-1]+(z+a)G \quad (3.31)$$

and

$$\Omega_{-a} = \frac{E}{K}\{e^{-(z+a)\gamma}-1\}\left\{1+\frac{i\alpha(\eta-\nu)}{2A^2}+O(\eta+\nu)^2\right\}-\frac{i\alpha E}{A}(z+a)+O(z+a)^2, \quad (3.32)$$

where D, E, F, G are functions of r only.

We now expand h_1 and Ω_1 ,about $z = a$, letting $(a-z) = \epsilon_1$. Hence from (3.24) we get

$$h_1(a-\epsilon_1) = \frac{J_1(\mu r)}{r}\{B\sin\beta a+C\cos\beta a+\beta\epsilon_1(-B\cos\beta a+C\sin\beta a)+O(\epsilon_1^2)\}, \quad (3.33)$$

and

$$\Omega_1(a-\epsilon_1) = \frac{iJ_1(\mu r)}{r}\left[(-B\cos\beta a+C\sin\beta a)\left\{1+\frac{i(\nu-\eta)}{2A^2\alpha}(\mu^2A^2+\alpha^2)\right\}\right.$$
$$\left.-\epsilon_1\beta(B\sin\beta a+C\cos\beta a)+O(\eta-\nu)\epsilon_1+O(\eta+\nu)^2\right] \quad (3.34)$$

when ϵ_1 is small.

For a similar expansion of h_1, Ω_1 about $z = -a$ we let $(z+a) = \epsilon_2$. Hence

$$h_1(-a+\epsilon_2) = \frac{J_1(\mu r)}{r}\{-B\sin\beta a+C\cos\beta a+\epsilon_2\beta(B\cos\beta a+C\sin\beta a)+O(\epsilon_2^2)\}, \quad (3.35)$$

and

$$\Omega_1(-a+\epsilon_2) = \frac{iJ_1(\mu r)}{r}\left[(-B\cos\beta a-C\sin\beta a)\left\{1+\frac{i(\nu-\eta)}{2A^2\alpha}(\mu^2A^2+\alpha^2)\right\}+O(\eta+\nu)^2\right.$$
$$\left.+\beta\epsilon_2(-B\sin\beta a+C\cos\beta a)+O(\nu-\eta)\epsilon_2\right]. \quad (3.36)$$

We now let $z \to -\infty$ (on the boundary-layer scale) to obtain

$$\left.\begin{aligned} -D+F\epsilon_1 &= h_1(a-\epsilon_1),\\ \frac{D}{K}\left\{1+\frac{i\alpha(\eta-\nu)}{2A^2}\right\}+\frac{i\alpha D\epsilon_1}{A} &= \Omega_1(a-\epsilon_1),\end{aligned}\right\} \quad (3.37)$$

and in the boundary-layer solutions at $z = -a$ let $z \to \infty$ ($z = O(\eta\nu)^{\frac{1}{2}}$) to give

$$\left.\begin{aligned} -E+G\epsilon_2 &= h_1(-a+\epsilon_2), \\ -\frac{E}{K}\left\{1+\frac{i\alpha(\eta-\nu)}{2A^2}\right\}-\frac{i\alpha E\epsilon_2}{A} &= \Omega_1(-a+\epsilon_2). \end{aligned}\right\} \tag{3.38}$$

From the matching conditions (3.37) and (3.38) we obtain for a non-trivial solution

$$e^{2i\beta\gamma} = \pm\left(\frac{1-K}{1+K}\right)\left\{1-\frac{i\sqrt{(\eta\nu)}\,(\mu^2A^2+\alpha^2)}{A^2\alpha}+O(\eta+\nu)^2\right\}. \tag{3.39}$$

Subsequently (3.25) and (3.39) become

$$\begin{aligned} \alpha = \frac{A}{2a}\left(m\pi - i\ln\left|\frac{1-K}{1-K}\right|\right)+\tfrac{1}{2}i(\eta+\nu)\left\{\mu^2+\frac{1}{4a^2}\left(m\pi-i\ln\left|\frac{1-K}{1+K}\right|\right)^2\right\} \\ +\sqrt{(\eta\nu)}\left\{\mu^2+\frac{1}{2a^2}\left(m\pi-i\ln\left|\frac{1-K}{1+K}\right|\right)^2\right\}\Big/\left(m\pi-i\ln\left|\frac{1-K}{1+K}\right|\right)+O(\eta+\nu)^2, \end{aligned} \tag{3.40}$$

which is the same value as was obtained from the exact solution.

4. Eigenvalues of the spherical problem

For the primary oscillations in r we assume for η and ν zero that the oscillations occur only on the axis of symmetry of the sphere, which is consistent with Stewartson's (1957) related problem. Stewartson's problem differs from the present one only in the boundary conditions. The additional boundary condition is satisfied by a boundary layer at $z = \pm a$ as in the cylindrical case (figure 2(*b*)). Consequently the leading term ($\eta = \nu = 0$) for the eigenvalues of the spherical problem is the same as the corresponding term of the cylindrical case, namely α as given by equation (3.22).

We use the same approach as in the cylindrical case to determine the first-order effect of η and ν. First, we assume that in the interior of the fluid the governing equation is similar to that for the cylindrical case (equation 3.23), i.e.

$$\left\{\alpha^2+i(\eta+\nu)\,D+A^2\frac{\partial^2}{\partial z^2}\right\}h_1 = O(\eta\nu h_1), \tag{4.1}$$

where
$$D = \frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

Let us consider the first-order effect of curvature in the immediate vicinity of the axis of symmetry by letting

$$z = \xi(1-\tfrac{1}{2}y^2),$$

where $y = r/a \ll 1$. The boundary conditions are $\Omega = h = 0$ at $z = \pm a$, h and $\Omega \to 0$ as y moves away from the axis.

Upon retaining leading terms in r, equation (4.1) becomes

$$\left\{\alpha^2+\frac{i\alpha(\eta+\nu)}{a^2}\left(\frac{\partial^2}{\partial y^2}+\frac{3}{y}\frac{\partial}{\partial y}\right)+A^2(1+y^2)\frac{\partial^2}{\partial \xi^2}\right\}h_1 = O(\eta\nu D^2 h_1) \tag{4.2}$$

in terms of the new variables ($\partial/\partial y \gg \partial/\partial\xi$).

The solution of (4.2) is of the form

$$h_1 = Y(y)\,(B\sin\beta\xi + C\cos\beta\xi) \tag{4.3}$$

with
$$\frac{d^2Y}{dy^2}+\frac{3}{y}\left(\frac{dY}{dy}\right)+\frac{a^2}{i\alpha(\eta+\nu)}\{\alpha^2-\beta^2(1+y^2)A^2\}\,Y = 0. \tag{4.4}$$

If we let
$$\mu = \left\{\frac{\alpha^2-\beta^2A^2}{i\alpha(\eta+\nu)}\right\}a^2, \quad \delta = \frac{\beta^2A^2a^2}{i\alpha(\eta+\nu)} \quad \text{and} \quad y = \delta^{-\frac{1}{4}}t^{\frac{1}{2}}, \tag{4.5}$$

we get from (4.4)
$$t\frac{d^2Y}{dt^2}+2\left(\frac{dY}{dt}\right)+\frac{1}{4}\left(\frac{\mu}{\sqrt{\delta}}-t\right)Y \simeq 0. \tag{4.6}$$

A solution is possible (Stewartson 1957) only if

$$\tfrac{1}{4}\mu/\sqrt{\delta} = n+1, \tag{4.7}$$

where n is an integer. The solution is then

$$Y = e^{-\frac{1}{2}t}\sum_{k=0}^{n}\frac{(-)^k\,n!\,t^k}{(n-k)!\,k!\,(k+1)!}. \tag{4.8}$$

For finite values of n the disturbance falls off as $\exp\{-r^2\sqrt{(\eta+\nu)}\}$ and so is confined to the axis of symmetry. This is consistent with our initial assumption. The disturbance is negligible except in the region where $r^2 < O(\eta+\nu)^{\frac{1}{2}}$.

From (4.7) and (4.5) we obtain

$$\beta = \frac{\alpha}{A}\left(1-\frac{2(n+1)\sqrt{\{i\alpha(\eta+\nu)\}}}{a\alpha}+O(\eta+\nu)\right). \tag{4.9}$$

It follows that $D = O(\eta+\nu)^{-\frac{1}{2}}$ and hence equation (4.2) is satisfied to $O\{\eta\nu/(\eta+\nu)\}$. Furthermore, $Y(0) \neq 0$ which is required by §2 (i.e. $C(0) \neq 0$).

From (2.5) and (2.6) we have the solution

$$\Omega_1 = i(-B\cos\beta\xi + C\sin\beta\xi)$$
$$\times\left(1+\frac{2(K^2-1)(n+1)\sqrt{\{i\alpha(\eta+\nu)\}}}{(K^2+1)\,a\alpha}+O(\eta+\nu)+O(y^2)\right)Y(y), \tag{4.10}$$

where $K = \sqrt{(\nu/\eta)}$ and $y^2 < O(\eta+\nu)^{\frac{1}{2}}$.

In the boundary layer $\partial/\partial\xi = O(\eta\nu)^{-\frac{1}{2}}$; if we let $\bar{\xi}\sqrt{(\eta\nu)} = (\xi\pm a)$ in (2.5) and (2.6), we get

$$\frac{\partial}{\partial\bar{\xi}}\left(\frac{\partial^2}{\partial\bar{\xi}^2}-A^2\right)h = O(\eta\nu)^{\frac{1}{2}}. \tag{4.11}$$

Near $\sqrt{(\eta\nu)}\,\bar{\xi} = \xi - a$ we have

$$h_a = D(e^{\bar{\xi}A}-1)+O(\eta\nu)^{\frac{1}{2}}, \quad \Omega_a = \frac{D}{K}(1-e^{\bar{\xi}A})+O(\eta\nu)^{\frac{1}{2}}, \tag{4.12}$$

and near $\sqrt{(\eta\nu)}\,\bar{\xi} = \xi + a$

$$h_{-a} = E(e^{-\bar{\xi}a}-1)+O(\eta\nu)^{\frac{1}{2}}, \quad \Omega_{-a} = \frac{E}{K}(e^{\bar{\xi}A}-1)+O(\eta\nu)^{\frac{1}{2}}, \tag{4.13}$$

where D and E are functions of r only.

From the matching conditions

$$\left.\begin{aligned} h_a(-\infty) &= h_1(a), & \Omega_a(-\infty) &= \Omega_1(a), \\ h_{-a}(\infty) &= h_1(-a), & \Omega_{-a}(\infty) &= \Omega_1(-a), \end{aligned}\right\} \tag{4.14}$$

we obtain

$$e^{2i\beta a} = \pm\left(\frac{1-K}{1+K}\right)\left(1+\frac{4(n+1)K\sqrt{\{i\alpha(\eta+\nu)\}}}{(1+K)a\alpha}+\text{higher } O(\eta+\nu)\right). \quad (4.15)$$

Equation (4.15) reduces to

$$\alpha = \frac{A}{2a}\left(m\pi - i\ln\left|\frac{1-K}{1+K}\right|\right)+\frac{(n+1)(i+1)}{a^2(1+K^2)}\left\{Aa(\eta+\nu)\Big/\left(m\pi-i\ln\left|\frac{1-K}{1+K}\right|\right)\right\}^{\frac{1}{2}}$$
$$\times\left\{(1+K^2)\left(m\pi-i\ln\left|\frac{1-K}{1+K}\right|\right)-2Ki\right\}+\text{higher } O(\eta+\nu), \quad (4.16)$$

where $K \neq 1$.

Thus for η and ν small there is a doubly infinite discrete set of eigenvalues with the disturbances decaying logarithmically.

The case of $\eta = \nu$ can be solved exactly. Let $y = \Omega \pm h$ and then (2.5) and (2.6) may be written as

$$i\alpha y = \pm A\frac{\partial y}{\partial z}+\nu\left(\frac{\partial^2 y}{\partial r^2}+\frac{3}{r}\frac{\partial y}{\partial r}+\frac{\partial^2 y}{\partial z^2}\right), \quad (4.17)$$

where $y = 0$ at $R^2 = r^2+z^2 = a^2$. Introduce spherical co-ordinates (R,θ,ϕ), where

$$z = R\cos\theta, \quad r = R\sin\theta. \quad (4.18)$$

Then equation (4.17) becomes

$$i\alpha y = \pm A\left(\cos\theta\frac{\partial y}{\partial R}-\frac{\sin\theta}{R}\frac{\partial y}{\partial\theta}\right)+\nu\left(\frac{\partial^2 y}{\partial R^2}+\frac{4}{R}\frac{\partial y}{\partial R}+\frac{1}{R^2}\frac{\partial^2 y}{\partial\theta^2}+\frac{3\cot\theta}{R^2}\frac{\partial y}{\partial\theta}\right). \quad (4.19)$$

If we let

$$y = P(\theta)F(R)\exp(\mp\tfrac{1}{2}AR\cos\theta/\nu), \quad (4.20)$$

equation (4.19) becomes

$$\frac{d^2P}{d\theta^2}+3\cot\theta\left(\frac{dP}{d\theta}\right)+CP = 0, \quad (4.21)$$

and

$$\frac{d^2F}{dR^2}+\frac{4}{R}\left(\frac{dF}{dR}\right)-\left(\frac{\nu C}{R^2}+i\alpha+\tfrac{1}{4}A^2/\nu\right)\frac{F}{\nu} = 0, \quad (4.22)$$

C being an arbitrary constant of separation.

The boundary conditions now become $F = 0$ when $R = a$. The solution of (4.22) is then

$$F(R) = J_\beta(\mu R)/R^{\frac{3}{2}}, \quad (4.23)$$

where

$$\left.\begin{aligned}\mu^2 &= -(i\alpha+\tfrac{1}{4}A^2/\nu)/\nu, \quad \mu a = k_{\beta n}(\text{real}),\\ \beta &= \tfrac{1}{2}\sqrt{(9+4C)} \geqslant \tfrac{3}{2},\end{aligned}\right\} \quad (4.24)$$

β = order of the Bessel function and $k_{\beta n}$ denotes the zeroes. In order to solve (4.22) let $\chi = \cos\theta$, then

$$(1-\chi^2)\frac{d^2P}{d\chi^2}-4\chi\frac{dP}{d\chi}+CP = 0. \quad (4.25)$$

The solutions then are

$$P_1(\chi) = \chi+2\sum_{k=1}^{k=m}\left[\frac{(-)^k(m-1)!}{(2k+1)!(m-k-1)!}\{(2m+3)(2m+5)\dots(2m+2k+1)\}\right]\chi^{2k+1}, \quad (4.26)$$

where m is an integer and, $C_1 = 2(2m-1)(m+1)$, $\beta_1 = 2m+\frac{1}{2}$, and

$$P_2(\chi) = 1+\sum_{k=1}^{k=m}\left[\frac{(-)^k(m-1)!}{(2k)!(m-k-1)!}\{(2m+1)(2m+3)\dots(2m+2k-1)\}\chi^{2k}\right], \quad (4.27)$$

where $C_2 = 2(m-1)(2m+1)$, $\beta_2 = 2m-\frac{1}{2}$. Hence Ω and h may be written as

$$\Omega = \frac{e^{-\frac{1}{2}Aa/\nu}\cosh\left(\frac{1}{2}Az/\nu\right)}{R^{\frac{3}{2}}}\begin{Bmatrix} J_{2m+\frac{1}{2}}(\mu_1 R)\, P_1(\cos\theta), \\ J_{2m-\frac{1}{2}}(\mu_2 R)\, P_2(\cos\theta), \end{Bmatrix} \tag{4.28}$$

$$h = \frac{e^{-\frac{1}{2}Aa/\nu}\sinh\left(\frac{1}{2}Az/\nu\right)}{R^{\frac{3}{2}}}\begin{Bmatrix} J_{2m+\frac{1}{2}}(\mu_1 R)\, P_1(\cos\theta'), \\ J_{2m-\frac{1}{2}}(\mu_2 R)\, P_2(\cos\theta), \end{Bmatrix} \tag{4.29}$$

where $m = 1, 2, 3, \ldots,$

$$\left.\begin{aligned} a\mu_1 = k\beta_1 n, \quad a\mu_2 = k\beta_2 n, \\ \alpha = i\{\tfrac{1}{4}(A^2/\nu) + \mu_{1,2}^2 \nu\}. \end{aligned}\right\} \tag{4.30}$$

Thus for the special case of $\eta = \nu$ the disturbances are confined to the vicinity of the two points on the spherical surface intersected by the axis of symmetry. In the limit of $\nu \to 0$ the disturbance is confined to two points, $z = \pm a$, on the spherical surface.

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